Abstract

By a simple trick we may generalise the rook polynomial for an \( n \times n \) chessboard to various 2-dimensional surfaces, the conventional chessboard corresponding to the torus. In the case of the Möbius band and the Klein bottle, there is a close connection to graceful labellings of graphs.

1 Introduction

Let \( S_n \) denote the group of permutations of \( [n] = \{1, \ldots, n\} \). For a square matrix \( A = [a_{ij}] \) of order \( n \), the permanent of \( A \), denoted \( \text{per} A \), is defined by

\[
\text{per} A = \sum_{\alpha \in S_n} \prod_{i=1}^{n} a_{\alpha(i),i}.
\]

For an \( m \times n \) matrix \( A = [a_{ij}] \), with \( m \leq n \), and for subsets of \( X \) and \( Y \) of \( [m] \) and \( [n] \), respectively, we denote by \( A[X|Y] \) the sub-matrix of \( A \) consisting of the rows indexed by \( X \) and the columns indexed by \( Y \). Then we extend the definition of the permanent to the non-square matrix \( A \), by

\[
\text{per} A = \sum_{Y \subseteq [n], |Y| = m} \text{per} A[m|Y].
\]

Following [4], we define

\[
\sigma_k(A) = \begin{cases} 
1 & k = 0 \\
0 & k > m \\
\sum_{X,Y} \text{per} A[X|Y] & 1 \leq k \leq m
\end{cases}
\]
where the summation runs over \( k \)-subsets of \( X \) of \([m]\) and \( k \)-subsets \( Y \) of \([n]\).

If \( A \) is a \((0, 1)\)-matrix then it may be identified with an \( m \times n \) chessboard in which squares corresponding to zeros in \( A \) have been ‘deleted’, and upon which we are to place rooks, which ‘attack’ along the row and column of the square they inhabit. Then \( \sigma_k(A) \) specifies the number of ways that \( k \) mutually non-attacking rooks may be placed on the chessboard. In particular, \( \sigma_n(A) = \text{per} A \) is the number of ways to place a rook on each row of the chessboard, with no rook attacking any other. The vector \( r_A = [\sigma_m(A), \ldots, \sigma_0(A)] \) is called the rook vector of \( A \) and the polynomial \( r(A; x) = \sum \sigma_i(A)x^i \) is the rook polynomial of \( A \). If no squares are deleted then \( A \) is the \( m \times n \) all-ones matrix which we denote \( J_{m,n} \), or \( J_n \) in the case where \( m = n \).

In this paper we shall largely confine ourselves to \( n \times n \) chessboards with no deleted squares but we shall generalise in a different way by identifying pairs of opposite edges of the chessboard in a manner familiar from the topology of two-dimensional surfaces ([3] gives a nice overview.) At the same time we draw a boundary line down the diagonal of the chessboard and decree that rooks cannot attack each across this boundary. Fig. 1 shows the five possible surfaces: fig. 1(a) shows the cylinder together with a placement which is legal only for the cylinder; fig. 1(b) shows a placement which is legal only for the Möbius band; fig. 1(c) shows a placement which is legal for both the torus and cylinder; fig. 1(d) shows a placement which is legal for the Klein bottle, Möbius band and cylinder; and in fig. 1(e), the projective plane is shown with the only possible legal placement of four rooks. This placement is also legal for each of the four previous boards.

It is immediately apparent that the rook polynomial of the torus is equivalent to what we may call the ‘classical’ definition of the rook polynomial, since the identification of the sides of the chessboard render the diagonal barrier irrelevant. The rook polynomial in this case, deriving from the matrix \( J_n \), is well-known:

\[
    r(J_n; x) = \sum_{k=0}^{n} \frac{n!}{k!} \left( \frac{n}{k} \right)^2 x^k
\]

We further observe that the rook polynomials for the cylinder and Möbius band can be analysed in ‘classical’ terms since both chessboards maybe unfolded into a rectangular chessboard with deleted squares, illustrated in fig. 2 for the case \( n = 4 \).
We shall analyse the rook polynomials associated with the five two-dimensional surfaces in the order suggested by fig. 1. The Möbius band and torus prove, via fig. 2(b), to have identical rook polynomials (they are rook-equivalent) and in this case we have a more or less complete solution. However, for the other boards we are only able to give the leading term for the associated polynomial, leaving unsolved what appear to be some intriguing questions. For instance, the leading coefficient for the cylinder is given as an untidy double sum of products but it appears that one might be able to express this as the reversion of an exponential generating function. The leading coefficient in the case of the Klein bottle is found via a connection with graceful labellings of graphs: we exhibit two matrices of indeterminants having the property that the determinant of their difference enumerates graceful labellings of trees, while the permanent of their sum enumerates $n$-rook placements on the $n \times n$ Klein bottle. A more complete understanding of this case might shed light on the notorious graceful tree conjecture [5]. Finally the leading coefficient of the projective plane board is seen to be unity, by a simple inductive argument but there is no reason to suppose that the other coefficients are so simple: it may be that they share with the classical rook polynomials of the $n \times n$ chessboard the property that they solve some differential equation: we pose this as our final question.

2 The cylinder

In this and the next section we shall draw repeatedly on the useful paper of Cheon et al [4], providing connections between rook polynomials and permanents.

Let $R_n = [r_{ij}]$ be the $n \times 2n - 1$ matrix defined by

$$r_{ij} = \begin{cases} 1 & i \leq j \leq n + i - 1 \\ 0 & \text{otherwise} \end{cases},$$

and let $L_n = [l_{ij}]$ be the $n \times n$ lower triangular matrix defined by

$$l_{ij} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise} \end{cases}.$$
Recall that the $m \times n$ Ferrers matrix, $F_n(b_1, \ldots, b_m)$, has $m$ rows in which the $i$-th row consists of $b_i$ ones, followed by $n - b_i$ zeros, with $b_i \leq b_{i+1}$, $i = 1, \ldots, m-1$ and $b_m = n$. We have

$$\text{per } F_n(b_1, \ldots, b_m) = \prod_{i=1}^{m} (b_i - i + 1).$$  \hfill (3)

We observe that $L_n$ is a Ferrers matrix and define a $2n - 1 \times 2n - 1$ matrix $\tilde{R}_n$ by adding $n - 1$ rows of ones above $R_n$:

$$\tilde{R}_n = \begin{bmatrix} J_{n-1} & J_{n-1,n} \\ L_{n-1}^T & 0 \end{bmatrix}.$$

It is easy to see that

$$\text{per } R_n = \frac{1}{(n-1)!} \text{per } \tilde{R}_n.$$  \hfill (4)

Now Cheon et al have the following result:

**Theorem 1** ([4]) Let $X$ be a nonnegative square matrix of order $m + n$ partitioned as

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A$ and $D$ are nonzero square matrices of order $m$ and $n$, respectively, $m \leq n$. Then

$$\text{per } X = \sum_{r=0}^{m} \sum_{\alpha,\beta,\gamma,\delta} \text{per } A[\bar{\alpha}|\bar{\beta}] \text{ per } D[\bar{\gamma}|\bar{\delta}] \text{ per } B[\alpha|\delta] \text{ per } C[\gamma|\beta],$$  \hfill (5)

where the second summation runs over all $r$-subsets $\alpha, \beta$ of $[m]$ and $r$-subsets $\gamma, \delta$ of $[n]$, and empty sub-matrices in the summation are taken to have permanent unity.

Applying equations (4) and (5) to $\tilde{R}_n$, we obtain

$$\text{per } R_n = \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \sum_{\alpha,\beta,\gamma,\delta} \text{per } J_{n-1}[\bar{\alpha}|\bar{\beta}] \text{ per } L_n[\bar{\gamma}|\bar{\delta}] \text{ per } J_{n-1,n}[\alpha|\delta] \text{ per } L_{n-1}^T \left[ \begin{array}{c} L_{n-1}^T \\ 0 \end{array} \right] [\gamma|\beta]$$  \hfill (6)

Subsets $\bar{\alpha}$ and $\bar{\beta}$ both have cardinality $n - 1 - r$ and $\alpha$ and $\delta$ have cardinality $r$. Then $\text{per } J_{n-1}[\bar{\alpha}|\bar{\beta}] = (n - 1 - r)!$ and $\text{per } J_{n-1,n}[\alpha|\delta] = r!$ and (6) becomes

$$\text{per } R_n = \sum_{r=0}^{n-1} \left( \begin{array}{c} n-1 \\ r \end{array} \right)^{-1} \sum_{\alpha,\beta,\gamma,\delta} \text{per } L_n[\bar{\gamma}|\bar{\delta}] \text{ per } L_{n-1}^T \left[ \begin{array}{c} L_{n-1}^T \\ 0 \end{array} \right] [\gamma|\beta].$$
\[
= \sum_{r=0}^{n-1} \sum_{\beta,\gamma,\delta} \text{per } L_n[\gamma|\delta] \text{ per } \begin{bmatrix} L_{n-1}^T \\ 0 \end{bmatrix} [\gamma|\delta]
\]
\[
= \sum_{r=0}^{n-1} \sum_{\gamma} \prod_{i=1}^r (\gamma_i - i + 1) \prod_{j=1}^r (n - r + j - \gamma_j) \quad \text{by (1) and (3)}.
\]

We have been unable to simplify this rather cumbersome summation. However, the resulting sequence of leading coefficients for the cylinder rook polynomial, which is given in table 1 at the end of the paper, coincides (with an offset of 2) with sequence A088789 of the Online Encyclopedia of Integer Sequences [7], defined as the series reversion [8] of the exponential generating function \(2x/(1 + e^x)\). We thus raise the first of a series of open questions:

**Question 1** Derive the series given by (7) as an exponential generating function and find an expression for the remaining terms of the cylinder rook polynomial.

### 3 The Möbius band and torus

The pyramid chessboard representing the Möbius band, illustrated in fig. 2(b), corresponds to an \(n \times 2n - 1\) \((0,1)\)-matrix \(P_n = [p_{ij}]\) defined by

\[
p_{ij} = \begin{cases} 1 & i > \max(n - j, j - n) \\ 0 & \text{otherwise} \end{cases}
\]

By column permutations (see [2, Theorem 4]) the rook vector of \(P_n\) is the same as the rook vector of the \(n \times 2n - 1\) Ferrers matrix \(F_{2n-1}(1,3,\ldots,2n-1)\), (ie. \(F_{2n-1}(b_1,\ldots,b_n)\) with \(b_i = 2i - 1, i = 1,\ldots,n\).) Cheon et al [4] give a convenient method for calculating rook vectors of Ferrers matrices and we will use this calculation to confirm that the rook polynomials for the torus and Möbius band are the same.

For an \(n \times n\) Ferrers matrix \(A = F_n(b_1,\ldots,b_n)\), write

\[
f_A(k) = \prod_{i=1}^n (k + b_i - i + 1), \quad k = 0,\ldots,n.
\]

Let \(\Delta_n\) be the \(n+1 \times n+1\) matrix of binomial coefficients:

\[
\Delta_n = \begin{bmatrix}
\binom{0}{0} & 0 & 0 & \cdots & 0 \\
\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} 
\end{bmatrix},
\]
and let \( D_n = \text{diag}(0!, 1!, \ldots, n!) \). Then:

**Theorem 2 ([4])** The rook vector \( r_A \) of the Ferrers matrix \( A = F_n(b_1, \ldots, b_n) \) is given by

\[
r_A = D_n^{-1} \Delta_n^{-1} (f_A(0), \ldots, f_A(n))^T.
\]

It is observed, in [4], that \( \Delta_n^{-1} \) is given by

\[
\Delta_n^{-1} = \begin{bmatrix}
(0) & 0 & 0 & \cdots & 0 \\
-(1) & (1) & 0 & \cdots & 0 \\
(2) & -(2) & (2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^n(n) & (-1)^{n+1}(n) & (-1)^{n+2}(n) & \cdots & (n) \\
\end{bmatrix}
\]

(9)

We have seen that the rook vector for the pyramid chessboard is the same as that of \( F_{2n-1}(1, 3, \ldots, 2n-1) \). If this \( n \times 2n - 1 \) matrix is extended to a \( 2n-1 \times 2n-1 \) matrix \( A \) by adding \( n-1 \) rows of zeros at the top, then it can be calculated from (8) that

\[
f_A(k) = \begin{cases} 
0 & k = 0, \ldots, n-2 \\
\frac{k!(k+1)!}{(k-n+1)!^2} & k = n-1, \ldots, 2n-1 
\end{cases}
\]

(10)

Now theorem 2, together with (9) and (10) gives

\[
r_{n-k}(A) = \frac{1}{(n+k-1)!} \sum_{i=0}^{k} (-1)^{i+k} \binom{n+k-1}{k-i} \frac{(n+i-1)!(n+i)!}{(i!)^2}
\]

\[
= \sum_{i=0}^{k} \frac{(-1)^{i+k} \binom{k}{i} (n+i)!}{i!k!} 
\]

(11)

We now calculate the rook vector of the classical chessboard (the torus) another way and find it is identical to (11). First some more machinery from [4]: let \( O_{p,q} \) denote the \( p \times q \) zero matrix. We extend an \( m \times n \) matrix \( A, m \leq n \), to an \( n \times n \) matrix \( \tilde{A} \) by adding \( n-m \) rows of zeros:

\[
\tilde{A} = \begin{bmatrix}
A \\
O_{n-m,n}
\end{bmatrix}
\]

and define matrices \( Y_k(A) \) by \( Y_0(A) = \tilde{A} \) and

\[
Y_k(A) = \begin{bmatrix}
\tilde{A} & J_{n,k} \\
J_k & J_k
\end{bmatrix}, \quad k = 1, \ldots, n.
\]

Let \( y_k(A) \) denote per \( Y_k(A) \), \( k = 0, \ldots, n \). Then:
Theorem 3 ([4]) For an \( m \times n \) matrix \( A \), we have

\[
\sigma_{n-k}(A) = \sum_{i=0}^{k} \frac{(-1)^{i+k}}{i!k!} \binom{k}{i} y_i(A), \quad k = 0, \ldots, n.
\]

Comparing this to (11) it remains to observe that, for the classical \( n \times n \) chessboard, \( A = J_n \) and \( y_i(A) = (n + i)! \) and we have:

Theorem 4 The torus and Möbius band are rook-equivalent.

There is a simple bijection between placements of \( n \) rooks on the torus and on the Möbius band: observe in fig. 1(b) that the attacking lines of the rooks, along columns, partition the columns into \( n \) sets as illustrated in fig. 3(a). By the pigeon hole principle, an \( n \)-rook placement will involve placing one rook in each column partition set. There is exactly one choice in the set labelled \( n \); this leaves two choices out of the three for set \( n - 1 \); now we have three choices out of the four for set \( n - 2 \), and so on. In this way we generate exactly \( n! \) rook placements, in one-to-one correspondence with the \( n! \) placements of \( n \) rooks on the torus. However, we have been unable to extend this correspondence to answer the following:

Question 2 Find a bijection between the placements of \( n - k \) rooks on the torus and Möbius band, for \( k < n \) (thus giving a bijective proof of theorem 4.)

Another way of looking at the placement of \( n \) rooks is to label the cells of the Möbius band with a coordinate pair in such a way that the difference between the ordinates specifies the column partition set, as shown in fig. 3(b). In this case we see that each placement of \( n \) rooks corresponds to a graceful graph, that is, a graph on \( n \) edges in which vertices are labelled 1 to \( n + 1 \) and each edge is labelled with the difference of its end nodes, in such a way that the edges are labelled 1 to \( n \). The rook placements on the Möbius band demonstrate that there are exactly \( n! \) graceful graphs on \( n \) edges, a fact first observed by Sheppard [6].
4 The Klein bottle and graceful labellings

We continue with the theme of graceful labellings by transferring the coordinates of fig. 3(b) to the Klein bottle, as shown in fig. 4. Whereas placing \( n \) rooks on the \( n \times n \) Möbius band generates all graceful graphs, the Klein bottle restricts attention to \( n \)-edge spanning subgraphs of the complete graph on \( n + 1 \) vertices because only one rook is allowed in each row, so that there is an edge incident with each vertex (vertex \( n + 1 \) is incident with the edge \((1, n + 1)\) by virtue of the fact that the top left-hand cell must contain a rook.)

Spanning subgraphs of the complete graph \( K_{n+1} \) having \( n \) edges either contain a cycle or are spanning trees. In fact, it is clear that, as in fig. 4, the coordinates in each position may be interpreted as directed edges, so that a rook placement corresponds to a spanning subgraph of \( K_{n+1} \), oriented so that each vertex, other than \( n + 1 \), has outdegree one and the edges not contained in cycles are directed towards vertex \( n+1 \). If the subgraph has a cycle then there will be two corresponding rook placements, one for each orientation of the cycle. The smallest examples occur when \( n + 1 = 7 \) in which there are two labelled subgraphs containing cycles, giving rise to four different rook placements. Two are shown in fig. 5.

Spanning trees are enumerated as a determinant evaluation in the so-called Matrix Tree Theorem and we shall adapt this to enumerate graceful trees and \( n \)-rook placements on the \( n \times n \) Klein bottle. For an oriented graph \( G = (V, E) \), with \( |V| = m \) and \( |E| = n \), the incidence matrix, \( B(G) = [b_{ij}] \), is the \( m \times n \) matrix whose rows and columns are indexed by \( V \) and \( E \), respectively, with the entry corresponding to vertex \( i \) and edge \( j \) being given by

\[
b_{ij} = \begin{cases} 
1 & \text{if edge } j \text{ is directed away from vertex } i \\
-1 & \text{if edge } j \text{ is directed towards vertex } i \\
0 & \text{if edge } j \text{ is not incident with vertex } i 
\end{cases}
\]

It is well-known that, given an \( m - 1 \times m - 1 \) sub-matrix \( X \) of \( B(G) \), with the columns of \( X \) corresponding to subset \( F \subseteq E \), the determinant of \( X \) satisfies

\[
\det X = \begin{cases} 
\pm 1 & \text{if the edges in } F \text{ form a tree} \\
0 & \text{otherwise.}
\end{cases}
\]
Figure 5: two rook placements corresponding to different orientations of the same graph.

Let $B_{(i)}$ and $B_{(j)}$ be copies of $B(G)$ with the $i$th and $j$th row deleted, respectively, and consider the product $B_{(i)}B_{(j)}^T$. We will apply the Cauchy-Binet theorem: for any two matrices, $A$, $m \times n$, and $B$, $n \times m$, with $m \leq n$,

$$\det AB = \sum_{X \subseteq [n], |X| = m} \det A[[m]|X] \det B[X|[m]].$$

For $B_{(i)}$ and $B_{(j)}$, the sum will count $\pm 1$ for each spanning tree of $G$ and zero for all other sub-matrices. With a little care over the signs we have:

**Theorem 5 (The Matrix Tree Theorem)** Any row and column deleted minor of $B(G)B(G)^T$ has determinant whose absolute value equals the number of spanning trees of $G$.

We shall enumerate graceful trees as a subset of the spanning trees of the complete graph, which for notational convenience we shall now take on $n$ vertices (so we shall eventually deal with $n-1$-rook placements.) Let $B_n$ be the $n \times \frac{1}{2}n(n-1)$ incidence matrix of the complete graph on $n$ vertices labelled $1, \ldots, n$. Let each edge be weighted according to the absolute value of the difference of its end-vertex labels and arrange the columns of $B_n$ in descending order of the weights of the corresponding columns. Thus, for $K_4$, $B_4$ would be specified (not uniquely) as the matrix:

$$
\begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 & -1 & 1 \\
-1 & 0 & -1 & 0 & 0 & -1
\end{pmatrix}
$$
Specify a second matrix $C_n$ of size $\frac{1}{2}n(n - 1) \times n$ as follows: the first row is $e_2$, the $n$-unit vector which is zero except for a one in the second position; rows two and three are $e_3$, and so on, with rows $\frac{1}{2}k(k - 1) + 1, \ldots, \frac{1}{2}k(k + 1)$ being $k$ copies of $e_{k+1}$, for $k = 1, \ldots, n - 1$. Now let $W_n$ be a diagonal matrix of indeterminants with the diagonal entry corresponding to edge $ij$ in the ordering of the columns of $B_n$ being $x_{ij}$, and consider the product, $\Psi_n(x_{ij}) = B_n W_n C_n$, which, for $n = 4$ will be the product

$$
\begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 & -1 & 1 \\
-1 & 0 & -1 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
x_{14} & 0 & 0 & 0 & 0 \\
x_{13} & 0 & 0 & 0 & 0 \\
x_{24} & 0 & 0 & 0 & 0 \\
x_{12} & 0 & 0 & 0 & 0 \\
x_{23} & 0 & 0 & 0 & 0 \\
x_{34} & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Now $n - 1 \times n - 1$ sub-matrices of $C_n$ will be nonsingular if and only if they omit the first column and moreover correspond to subsets of edges of $K_n$ having different edge weights. So applying the Cauchy-Binet theorem we have:

**Theorem 6** Let $\Psi'_n(x_{ij})$ denote $B_n W_n C_n$ with its first row and column deleted. The multivariate polynomial in $x_{ij}$ given by $\det \Psi'_n(x_{ij})$ enumerates all gracefully labelled trees on $n$ vertices.

To continue the $K_4$ example, the determinant $\det \Psi'_4(x_{ij})$ will yield

$$-x_{14}x_{24}x_{23} + x_{14}x_{24}x_{34} - x_{14}x_{13}x_{12} + x_{14}x_{13}x_{23},$$

which is seen to enumerate the four graceful trees on four vertices and, simultaneously, the four placements of 3 rooks on the $3 \times 3$ Klein bottle.

The matrix $\Psi_n(x_{ij})$ has a rather ‘regular’ structure which we will now exploit to derive a permanent to count $(n - 1)$-rook placements on the Klein bottle. This structure is illustrated for $n = 4$:

$$\Psi_4(x_{ij}) = \begin{pmatrix}
0 & x_{14} & x_{13} & x_{12} \\
0 & 0 & x_{24} & -x_{12} + x_{23} \\
0 & 0 & -x_{13} & -x_{23} + x_{34} \\
0 & -x_{14} & -x_{24} & -x_{34}
\end{pmatrix}.$$

In general, we may define two $n \times n$ matrices, $\Upsilon_n(x_{ij}) = [\upsilon_{ij}]$ and $\Lambda_n(x_{ij}) = [\lambda_{ij}]$, by

$$\upsilon_{ij} = \begin{cases}
x_{i,n-j+i+1} & \text{if } i < j \\
0 & \text{if } i \geq j
\end{cases} \quad \text{and} \quad \lambda_{ij} = \begin{cases}
x_{j-n+i-1,i} & \text{if } i > n - j + 1 \\
0 & \text{if } i \leq n - j + 1
\end{cases}$$
and then $\Psi_n(x_{ij}) = \Upsilon_n(x_{ij}) - \Lambda_n(x_{ij})$.

Now the multivariate polynomial enumeration of graceful trees can be arranged as an alternating sum, so that setting all the indeterminants $x_{i}$ to one, which we denote by $x_{ij} \leftarrow 1$, would cause the determinant to disappear. This appears to prevent the use of $\Psi_n(x_{ij})$ as a counting function for graceful trees. It may be that some alternative matrix to $C_n$, having the same rank properties, might overcome this problem or at least allow lower bounds to be calculated for the number of graceful trees, in the spirit of the interesting work of Aldred et al [1].

If, on the other hand, we take instead the sum $\Upsilon_n(x_{ij}) + \Lambda_n(x_{ij})$ and take the permanent then the oppositely oriented cycles in spanning subgraphs of $K_n$, instead of cancelling out as in the Matrix Tree Theorem, are both included in the enumeration (a description of the cancellation which takes place is given by Zeilberger [9].)

**Theorem 7** If $\Phi_n(x_{ij})$ denotes the matrix $\Upsilon_n(x_{ij}) + \Lambda_n(x_{ij})$, and $\Phi'_n(x_{ij})$ denotes $\Phi_n(x_{ij})$ with first row and column deleted, then $\text{per}\Phi'_n(x_{ij})$ enumerates the ways to place $n - 1$ rooks on the $n - 1 \times n - 1$ Klein bottle. Moreover, $\text{per}\Phi'_n(x_{ij} \leftarrow 1)$ gives the leading term of the rook polynomial for the Klein bottle.

As a final example, we give the enumerating multivariate polynomial for $n = 5$:

\[
\begin{align*}
&x_{15}x_{25}x_{35}x_{45} - x_{15}x_{25}x_{35}x_{34} - x_{15}x_{25}x_{13}x_{45} + x_{15}x_{25}x_{13}x_{34} - x_{15}x_{25}x_{24}x_{23} \\
&+ x_{15}x_{25}x_{24}x_{34} + x_{15}x_{14}x_{24}x_{23} - x_{15}x_{14}x_{24}x_{34} - x_{15}x_{14}x_{12}x_{35} \\
&+ x_{15}x_{14}x_{13}x_{12} + x_{15}x_{14}x_{23}x_{35} - x_{15}x_{14}x_{13}x_{23},
\end{align*}
\]

which can be tested against fig. 4 to confirm that each term specifies a legal rook placement (remembering that edges are to be oriented towards vertex 5, so that $x_{14}$, say, will specify the 1,4 cell or the 4,1 cell of the Klein bottle, as required.)

To return briefly to the question of the signs of the terms in the graceful trees polynomial, we remark that these can be interpreted neatly from the trees themselves as follows: consider a gracefully labelled $n$-vertex tree $T$ with edges oriented towards vertex $n$. By virtue of the vertex labelling being graceful, $T$ defines a permutation $\pi_T$ in $S_{n-1}$: for each vertex $i$, other than $n$, $\pi_T$ maps $i$ to the unique edge label directed out of vertex $i$. For example, the term $+x_{15}x_{14}x_{24}x_{23}$, in the polynomial above, represents a tree with permutation (1 4 3). We may further define the sign of $T$ to be $(-1)^r$, where $r$ is the number of oriented edges $(i, j)$ in $T$ having $i < j$. The tree of the above term has sign $(-1)^2$. Then an examination of the correspondence between terms of $\det\Psi_n'(x_{ij})$ and graceful trees shows that

\[
\text{sign of term corresponding to } T \text{ in } \det\Psi_n'(x_{ij}) = \text{sign of } T \times \text{sign of } \pi_T.
\]

We are unable to resist asking the following, at the risk of seeming speculative:

**Question 3** What light, if any, can $\det\Psi_n'(x_{ij})$ shed on the graceful tree conjecture [5] that all trees admit a graceful labelling.
Figure 6: Projective plane with row and column partitions.

To return to issues closer to the present paper, the evaluation of $\Phi'_n(x_{ij} \leftarrow 1)$ in theorem 7 allows the leading term of the Klein bottle rook polynomial to be evaluated relatively easily as tabulated in table 1. We do not know of a way of evaluating the permanent involved in closed form. A larger question would seem to be:

**Question 4** *Determine the rook polynomials for the Klein bottle.*

A complete solution to the question should perhaps provide an analogue of Cheon et al’s formula, given in theorem 3.

### 5 The projective plane

As regards the projective plane, we will only observe that we can rather easily discover the leading term of the rook polynomial by repeating the column partitioning argument illustrated in fig. 3(a). The projective plane requires a simultaneous partitioning of the rows and columns as illustrated in fig. 6. In this case, to place $n$ rooks will require one to occupy each column partition set and each row partition set. This forces the top-left and bottom-right cells to be occupied and removes the first and last row and column. We then continue recursively, each step forcing the placement of two more rooks. We therefore see that, for even $n$, the leading coefficient of the rook polynomial of the projective plane is 1. The case for odd $n$ is exactly the same.

Apart from having spotted the obvious for its leading term, the following question is completely open:

**Question 5** *Calculate the rook polynomial for the projective plane.*

### 6 Conclusions

We have seen that the idea of a rook polynomial generalises in a simple way to the two dimensional surfaces which can be generated by identifying the edges of an $n \times n$ chessboard in various ways. The Möbius band polynomial does not differ
Table 1: number of ways of placing $n$ rooks on an $n \times n$ chessboard on the different 2-dimensional surfaces, together with the number of gracefully labelled trees on $n+1$ vertices. The latter is A033472 in [7] where it is given up to 22 vertices. The sequence for the cylinder appears to be identical to A088789($n+2$). The value for the projective plane is 1 for all $n$.

from the traditional rook polynomial (the torus) via this generalisation. However the cylinder and Klein bottle both give new polynomials for which only the leading term coefficients are derived in this paper, both being reduced to the calculation of the permanent of a certain integer matrix. The values of the leading coefficients are given in table 1, the projective plane being omitted because of its triviality.

There is an obvious connection between these new rook polynomials in that they all ‘belong’ to connected 2-dimensional surfaces. This connection might seem to lend an underlying topological framework to these generalised rook polynomials but there is no reason to suppose this has any significance. A more tangible link might be provided by looking for differential equations which are solved by the generalised rook polynomials. This idea is motivated by the Laguerre equation

$$x \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + ny = 0.$$ 

In the case where $n$ is confined to be a natural number, this has solutions defined
in terms of the Laguerre polynomials

\[
\begin{align*}
L_0(x) &= 1 \\
L_1(x) &= -x + 1 \\
L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\
L_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6),
\end{align*}
\]

and so on, the well-known relationship with equation 2 being clear. A final question might therefore be:

**Question 6** Are the rook polynomials for the cylinder, Klein bottle and projective plane similarly related to solutions of some associated differential equations?

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**References**


