

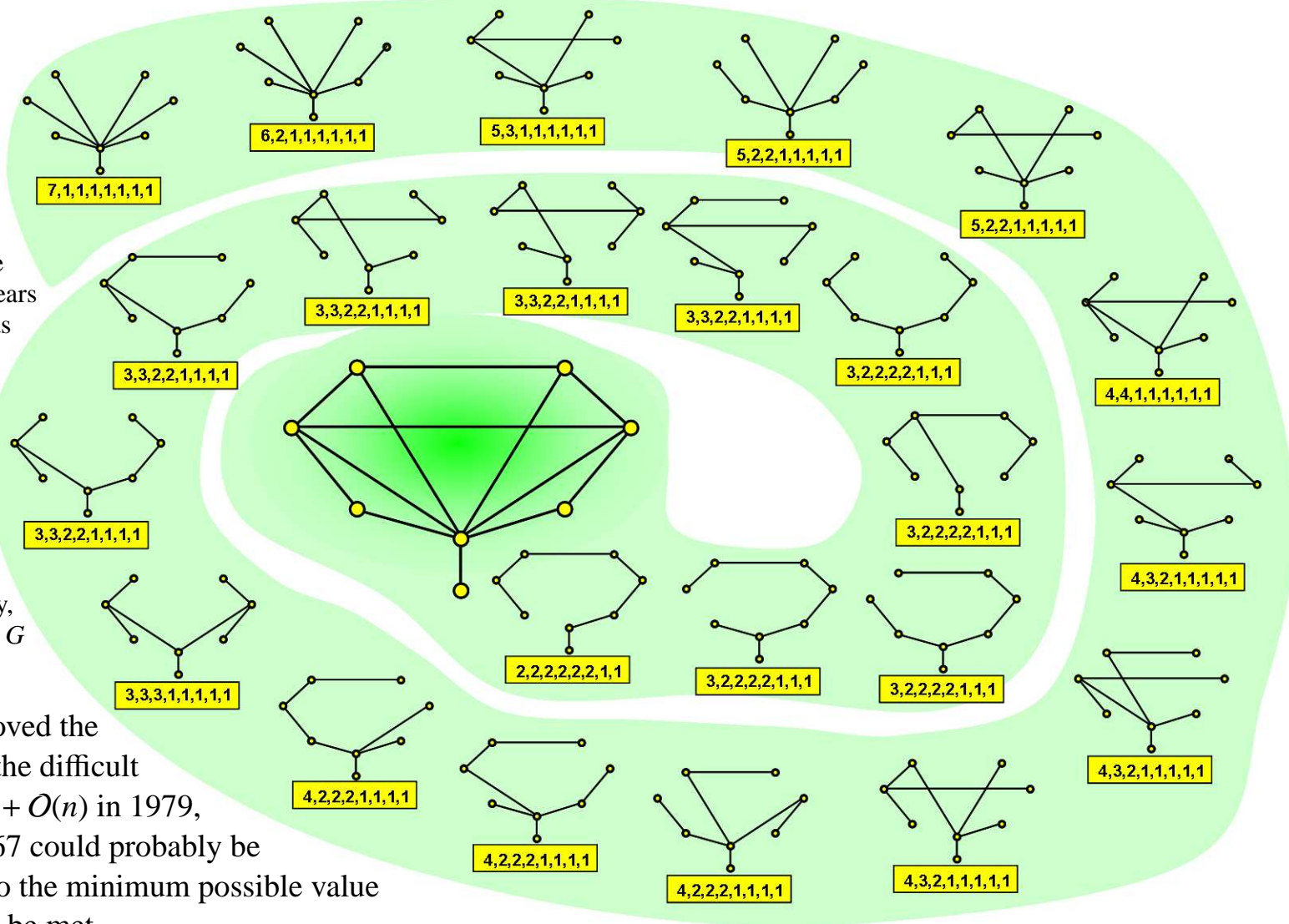


THEOREM OF THE DAY

The Panarboreal Formula Let \mathcal{T}_n denote the set of all unlabelled trees on n edges and denote by $s(\mathcal{T}_n)$ the minimum number of edges which an $(n+1)$ -vertex graph must have in order that it contains every tree in \mathcal{T}_n as a subgraph. Then $s(\mathcal{T}_n) \sim cn \log n$ where c is a constant satisfying $1/2 \leq c \leq 5/\log 4$.

There are 23 unlabelled trees having 7 edges; they are shown on the right, lexicographically ordered by degree sequence, together with an 8-vertex graph with 13 edges in which each one may be found as a subgraph. This appears to be the largest n for which $s(\mathcal{T}_n)$ has been calculated exactly; but it is easy to establish that $s(\mathcal{T}_n) \geq (1/2)n \log n$. For, given k , $1 \leq k \leq n+1$ we may always choose a tree in whose degree sequence the k -th entry $\geq n/k$. But now the same must hold for the degree sequence of any graph G containing this tree. So if G contains each tree in \mathcal{T}_n and has degree sequence, say, (d_1, \dots, d_{n+1}) , then number of edges in $G = \frac{1}{2} \sum_{k=1}^{n+1} d_k \geq \frac{1}{2} \sum_{k=1}^{n+1} n/k > \frac{1}{2} n \log n$.

Fan Chung and Ron Graham proved the easy lower bound on $s(\mathcal{T}_n)$ and the difficult upper bound of $(5/\log 4)n \log n + O(n)$ in 1979, mentioning that $5/\log 4 \approx 3.6067$ could probably be improved, possibly even down to the minimum possible value of $1/2$. This challenge has yet to be met.



Web link: math.ucsd.edu/~fan/ (where all Chung's papers may be found in pdf format)

Further reading: *Erdős on Graphs: His Legacy of Unsolved Problems*, by Fan Chung and Ronald Graham, AK Peters, 1998, section 3.5.1.

Theorem of the Day is maintained by Robin Whitty at www.theoremoftheday.org

