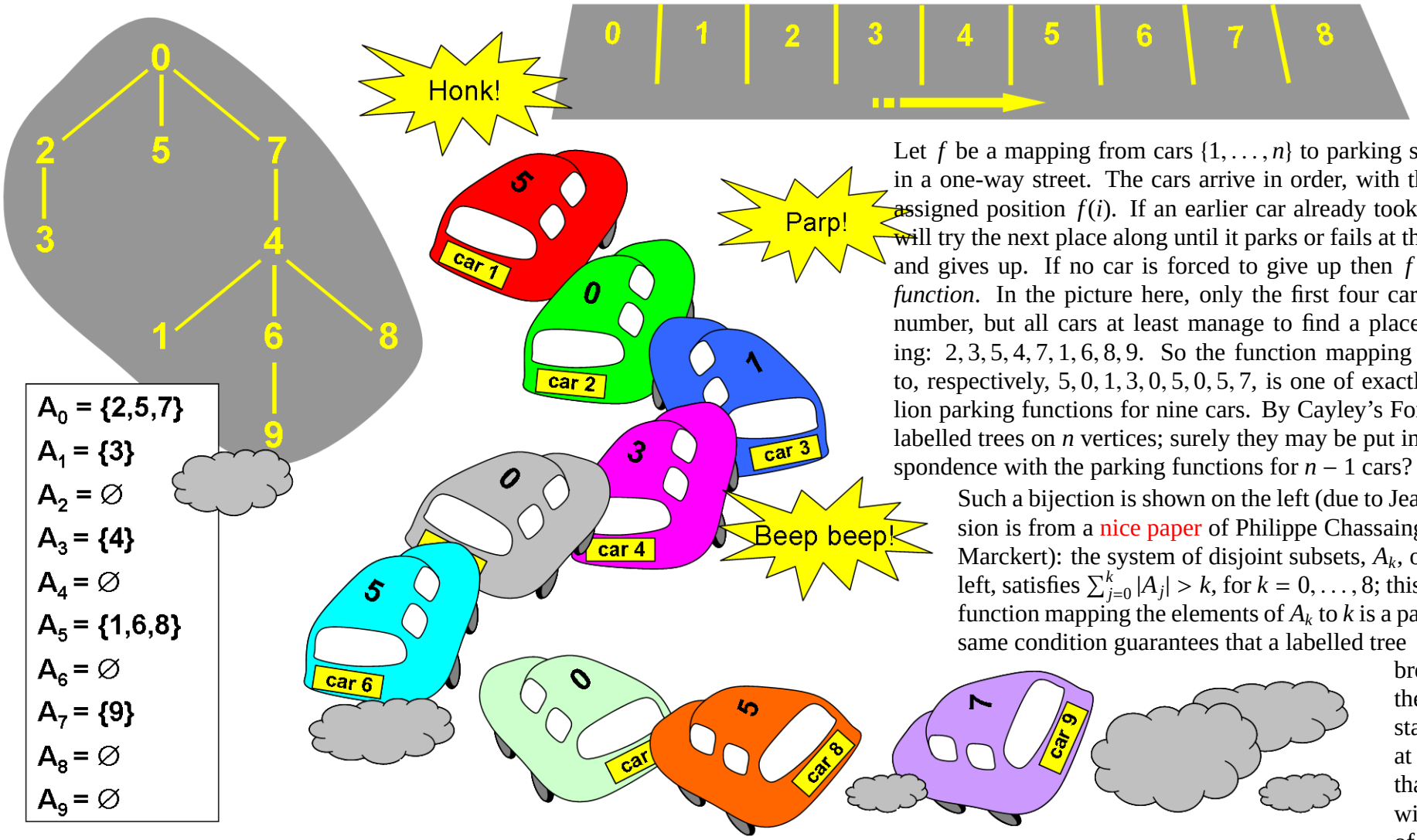
THEOREM OF THE DAY

The Parking Function Formula *The number of parking functions of order n is* $(n+1)^{n-1}$.



Let f be a mapping from cars $\{1,\ldots,n\}$ to parking spaces $\{0,\ldots,n-1\}$ in a one-way street. The cars arrive in order, with the i-th arrival being assigned position $f(i)$. If an earlier car already took that spot then car i will try the next place along until it parks or fails at the last parking space and gives up. If no car is forced to give up then f is called a *parking function*. In the picture here, only the first four cars get their intended number, but all cars at least manage to find a place, the allocation being: $2,3,5,4,7,1,6,8,9$. So the function mapping $1,2,3,4,5,6,7,8,9$ to, respectively, $5,0,1,3,0,5,0,5,7$, is one of exactly one-hundred million parking functions for nine cars. By Cayley's Formula there are n^{n-2} labelled trees on n vertices; surely they may be put into one to one correspondence with the parking functions for $n-1$ cars?

Such a bijection is shown on the left (due to Jean Françon; this version is from a nice paper of Philippe Chassaing and Jean-François Marckert): the system of disjoint subsets, A_k, of $\{0,\ldots,9\}$, bottom left, satisfies $\sum_{j=0}^{k}|A_j| > k$, for $k=0,\ldots,8$; this guarantees that the function mapping the elements of A_k to k is a parking function. The same condition guarantees that a labelled tree may be constructed breadth-first: build up the tree layer by layer, starting with vertex 0 at time 0; the vertex that is added at time t will have the elements of A_t added as children when the next layer is constructed.

Parking functions were introduced by Alan Konheim and Benjamin Weiss in the context of data storage but have come to be studied intensively as mathematical objects in their own right. The enumeration formula is proved in their original 1966 paper.

Web link: `www.maths.qmul.ac.uk/~pjc/preprints/art.pdf` is a nice glimpse of how professional mathematicians respond to stimulating ideas such as the above. And here is the result: `www.combinatorics.org/Volume_15/PDF/v15i1r92.pdf`.

Further reading: *Enumerative Combinatorics, Vol. 2,* by R.P. Stanley, CUP, 2001, Chapter 5 (exercises).