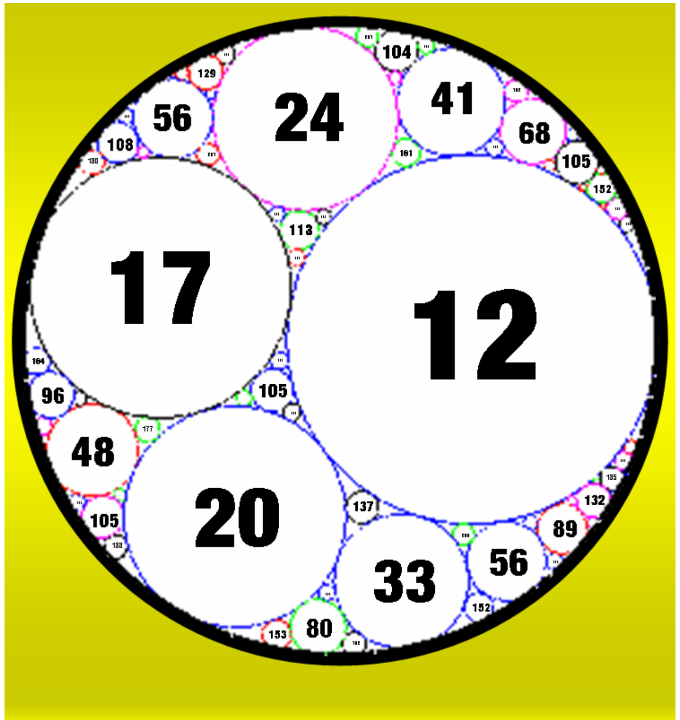
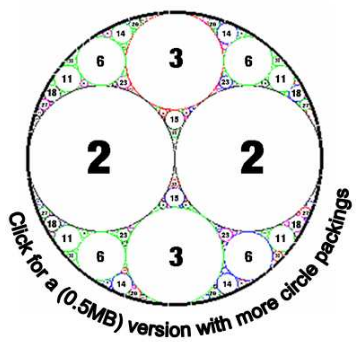
THEOREM OF THE DAY

A Theorem on Apollonian Circle Packings *For every integral Apollonian circle packing there is a unique 'minimal' quadruple of integer curvatures,* (a, b, c, d)*, satisfying* $a \leq 0 \leq b \leq c \leq d$*,* $a+b+c+d > 0$ *and* $a + b + c \geq d$*. This so-called* root quadruple *completely specifies the packing.*



(-7, 12, 17, 20)



A Descartes configuration consists of four mutually tangent circles. Above, for example, is a circle of radius 1/7 containing circles of radius 1/12, 1/17 and 1/20, each of which has a point of contact with the other three. The integers labelling the circles are the *curvatures* (the reciprocals of the radii) and in the root quadruple of curvatures, $(-7, 12, 17, 20)$, the enclosing circle of radius 1/7 is determined to have negative curvature so that all four circles have disjoint interiors. Any such configuration specifies four more tangent circles — above, these have curvatures 24, 33, 48, and 105, producing four new configurations $(-7, 12, 17, 24)$, $(-7, 12, 20, 33)$, $(-7, 17, 20, 48)$ and $(12, 17, 20, 105)$. Repeating this process produces a system of infinitely packed circles: an *Apollonian circle packing*. If our initial configuration is integral, as in the above example, then we will get an *integral* packing with every curvature an integer.

This theorem comes from a series of four pivotal papers by the AT&T team of Ronald Graham, Jeffrey Lagarias, Colin Mallows and Allan Wilks, together with Catherine Yan of Texas A&M University. They further show that all integral Apollonian circle packings may be derived from root quadruples which, like that depicted above, have entries whose gcd is 1.

Web link: www.vector.org.uk/archive/v224/cliff224.htm shows how to create Apollonian packings and links to the papers of Graham et al.

Further reading: *Introduction to Circle Packing: The Theory of Discrete Analytic Functions,* by Kenneth Stephenson, CUP, 2005.