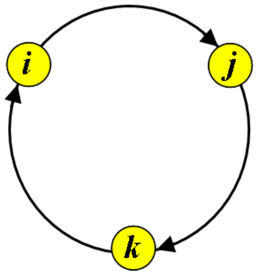
THEOREM OF THE DAY

Moufang's Theorem *In a Moufang loop any three elements which associate generate a group.*

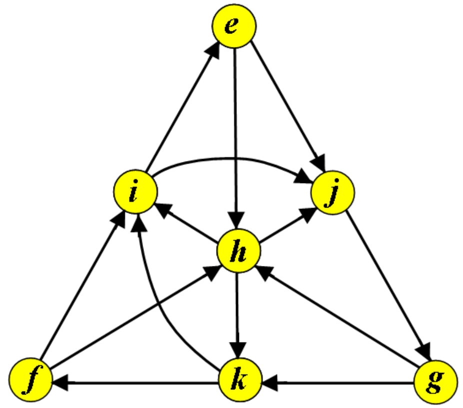
	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

The Quaternions



	1	i	j	k	e	f	g	h
1	1	i	j	k	e	f	g	h
i	i	-1	k	-j	f	-e	h	-g
j	j	-k	-1	i	-g	h	e	-f
k	k	j	-i	-1	h	g	-f	-e
e	e	-f	g	-h	-1	i	-j	k
f	f	e	-h	-g	-i	-1	k	j
g	g	-h	-e	f	j	-k	-1	i
h	h	g	f	e	-k	-j	-i	-1

The Octonions



Taking i to represent $\sqrt{-1}$, the set $\{\pm 1, \pm i\}$ forms a *group*: multiplication keeps you inside the set, it allows inverses (e.g. $i^{-1} = -i$, since $i \times -i = -(\sqrt{-1})^2 = -(-1) = 1$) and it is associative (that is, $x \times (y \times z)$ is the same as $(x \times y) \times z$ — the bracketing can safely be forgotten). In 1843, the great Irish scientist William Rowan Hamilton discovered the *quaternions*: i is joined by mysterious companions j and k who multiply according to the circular diagram above left: if x and y follow each other *clockwise* round the circle, then $x \times y = +$ the other quantity; if *anticlockwise*, the product is negative: $ij = k$, $kj = -i$, etc. And, again, $\{\pm 1, \pm i, \pm j, \pm k\}$ is a group. J.T. Graves, a professor of law in London, was inspired to try and go one better: just two months later he had produced the *octonions*, whose multiplication table is given centre and can be constructed from the Fano plane (above right; to keep the diagram simple, only three points from each circle are given: we must imagine $e \to j \to g$, for example, cycling back round to e). But Hamilton spotted a snag: octonion multiplication is not associative. For example, $(ij)e = ke = h$ but $i(je) = i(-g) = -ig = -h$. The octonions were discovered independently by Cayley and are sometimes called Cayley numbers.

A hundred years later, in Germany, Ruth Moufang invented a deep connection between algebra and projective geometry via the idea of a *loop*: exactly those arithmetics which fail to be groups just through being nonassociative. A *Moufang loop* is one in which any x, y and z *nearly* associate: they obey three (equivalent) *Moufang identities*:

$$\text{left: } (xy \cdot x)z = x(y \cdot xz), \qquad \text{middle: } (xy)(zx) = (x \cdot yz)x, \qquad \text{right: } (xy \cdot z)y = x(y \cdot zy).$$

You can check these hold in the octonions which are a classic example of a Moufang loop; the quaternions, hiding associatively inside, are a group thanks to Moufang's theorem. You can check, too, the corollary to Moufang's Theorem, that any Moufang loop is *diassociative*: any *pair* of elements whatsoever generates a group (put $y = 1$ in the left identity and apply the theorem to x, x and z).

Ruth Moufang (1905–1977) played an indirect part in the classification of the finite simple groups: Richard Parker's 1985 construction of a Moufang loop of order 2^{13} was used by John Conway to construct the Monster (order $\approx 8.10^{53}$).

Web link: emis.kaist.ac.kr/journals/CMUC/pdf/cmuc0201/Cheingoo.pdf

Further reading: *On Quaternions and Octonions* by J.H. Conway and D.A. Smith, AK Peters, 2003.