THEOREM OF THE DAY

The Total Probability Theorem *Suppose a sample space S is partitioned into n non-empty parts $B_1, B_2, \ldots, B_n$, $n \geq 1$. Then, for any event A,*

$$\mathbb{P}(A) = \sum \mathbb{P}(A \cap B_i) = \sum \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

This is a well-known illustration of how the total probability theorem is used in so-called 'next-step analysis'. Suppose that Federer wins each point independently with probability p, and that Nadal wins with probability $q = 1 - p$. What is the probability $\mathbb{P}(F) = v$, say, that Federer wins a game from deuce? The events F_1 and N_1 are clearly mutually exclusive and exhaustive — that is they partition S = {outcomes of next point}. So

$$\begin{aligned} v = \mathbb{P}(F) &= \mathbb{P}(F|F_1)\mathbb{P}(F_1) + \mathbb{P}(F|N_1)\mathbb{P}(N_1) \\ &= \mathbb{P}(F|F_1)p + \mathbb{P}(F|N_1)q. \qquad (1) \end{aligned}$$

Resolving the unknowns in equation (1) illustrates the extension of the theorem to conditional probabilities:

$$\begin{aligned} \mathbb{P}(F|F_1) &= \mathbb{P}(F|F_1 \cap F_2)\mathbb{P}(F_2|F_1) \\ &\quad + \mathbb{P}(F|F_1 \cap N_2)\mathbb{P}(N_2|F_1) \\ &= 1.p + v.q, \qquad (2) \end{aligned}$$

since the joint event $F_1 \cap F_2$ wins the game for Federer with probability 1 while $F_1 \cap N_2$ takes us back to deuce. For $\mathbb{P}(F|N_1)$ we get

$$\mathbb{P}(F|N_1) = v.p + 0.q. \qquad (3)$$

Sustituting (2) and (3) back into (1) gives us $v = (p + vq)p + vpq$. Rearranging and using $1 = p + q = (p + q)^2 = q^2 + 2pq + q^2$, we get Federer's chance of winning: $v = p^2/(p^2 + q^2)$, while Nadal's is $1 - v = q^2/(p^2 + q^2)$. These curves are plotted against each other above and demonstrate that the deuce rule has the effect of exaggerating the winning chances of the stronger player.

From the rules of tennis:

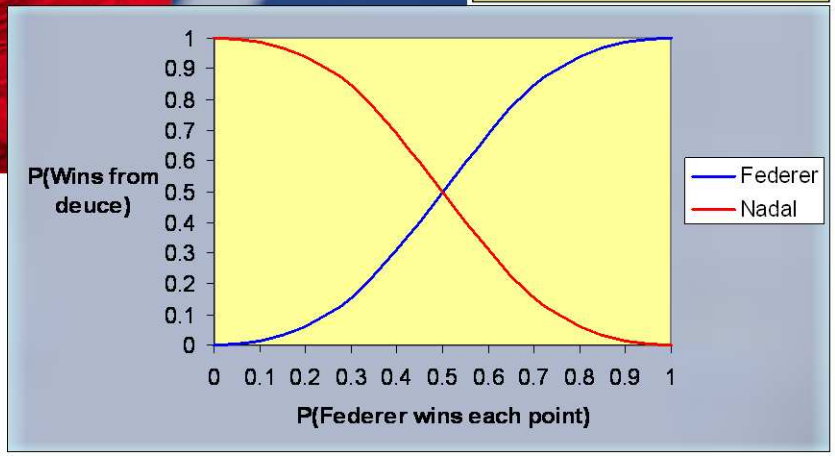
If the score is 40-40, also known as **deuce**, one side must win by two points.

Advantage-In means if the server wins the next point, he wins the game.

Advantage-Out means the receiver has a chance to win the game on the next point.

Events

F	=	Federer wins from deuce
F_1	=	Federer wins next point
N_1	=	Nadal wins next point
F_2	=	Federer wins point after next
N_2	=	Nadal wins point after next



The notions of conditional probability and of summing mutually exclusive probabilities appear in the writings of Bayes (1764).

Web link: www.stat.auckland.ac.nz/~stats325/notes.php, Chapter 2. Roger Federer image reproduced from the *New York Daily News* with the kind permission of www.dailynewspix.com. Deuce rule quoted from: westlake.k12.oh.us/hilliard/whspe/tennis/tennis_rules.htm.

Further reading: *Mathematical Statistics with Applications, 6th ed.* by D. Wackerly, W. Mendenhall III and R. Scheaffer, Duxbury, 2002, chapter 2.